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TECHNICAL NOTES

Forced convection heat transfer about an elliptic cylinder in a saturated porous medium

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1. INTRODUCTION

CONVECTIVE heat transfer by permeating fluids through a porous matrix has a wide range of applications. For instance heat flow studies in the hydrothermal systems of fractured rock formations are important in geothermal energy development [1–3]. The heat transport mechanism in granular materials is also important in insulation technology and various industrial operations such as reactors and absorbers [4, 5]. Several boundary layer solutions for forced convection such as over a flat plate, about a circular cylinder and a sphere are presently available [4, 6]. However, the parametric ranges of validity have not been properly identified; particularly the lower bounds of the Peclet number at which the transition from conduction to convection takes place. Therefore, the present note is intended to address this point for a particular convection system; namely forced convection about elliptic cylinders, which includes, as special cases, the heat transfer both over a flat plate and about a circular cylinder. In this note we first develop an integral solution for the heat transfer about elliptic cylinders and then clarify a range of its applicability solving the two-dimensional system of the Darcy model. Heat transfer about an elliptic cylinder is important, since this configuration occurs when a circular cylinder is subject to oblique flows. This problem has a potential application to geothermal well design.

2. INTEGRAL SOLUTION

We consider convective heat transfer about an isothermal cylinder as shown in Fig. 1. First assuming the presence of a thin thermal boundary layer along the cylinder surface, the energy conservation law in an integral form can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \{U(s)\,(T_* - T_\infty)\}\,\mathrm{d}\eta = -\alpha \left.\frac{\partial T_*}{\partial \eta}\right|_{\eta=0} \tag{1}$$

where s and η are the local coordinates as indicated in Fig. 1. Next introducing a temperature profile function $F(\Delta)$, its integral H and the thermal boundary layer thickness $\delta_T(s)$, equation (1) is put in a form

$$\frac{\mathrm{d}}{\mathrm{d}s}\{u(s)\delta_T(s)H\} = \frac{-\alpha F'(0)}{\delta_T(s)} \tag{2}$$

where

$$H = \int_0^1 F(\Delta)\,\mathrm{d}\Delta, \quad \Delta = \frac{\eta}{\delta_T}. \tag{3}$$

Equation (2) can be integrated once to yield

$$\delta_T(s) = \left\{\frac{-\alpha 2F'(0)}{H}\right\}^{1/2} \frac{\sqrt{(L(s))}}{U(s)} \tag{4}$$

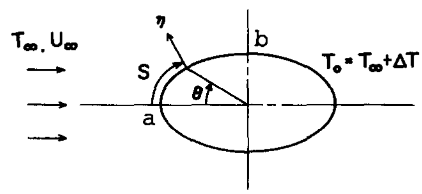


FIG. 1. The coordinate system for forced convection about an elliptic cylinder.

where

$$L(s) = \int_0^s U(s)\,\mathrm{d}s. \tag{5}$$

Applying the inviscid flow theory, the velocity along the elliptic arc is

$$\frac{U(\theta)}{U_\infty} = \frac{1+k}{\sqrt{(1+k^2\cot^2\theta)}} \tag{6}$$

where $k = b/a$. Therefore, the thermal boundary layer thickness is given by

$$\delta_T(\theta) = \left\{\frac{-2F'(0)}{H(1+k)}\cdot\frac{\alpha a}{U_\infty}\right\}^{1/2} \sqrt{((1-\cos\theta)\,(1+k^2\cot^2\theta))}. \tag{7}$$

The local heat transfer coefficient is

$$h(\theta) = -\frac{k_e}{\Delta T}\left.\frac{\partial T_*}{\partial \eta}\right|_{\eta=0} = k_e \left\{\frac{\dfrac{-HF'(0)\,(1+k)}{2}\cdot\dfrac{U_\infty}{a\alpha}}{(1-\cos\theta)\,(1+k^2\cot^2\theta)}\right\}^{1/2}. \tag{8}$$

The corresponding local Nusselt number is given by

$$Nu = \frac{h(\theta)S(\theta)}{k_e} \tag{9}$$

where the arc length $S(\theta)$ can be expressed in terms of the elliptic integral of the second kind. For instance if $\theta < \pi/2$

$$\begin{aligned} S(\theta) &= aE(\sqrt{(1-k^2)}, \pi/2-\theta)\big|_\theta^0 \quad \text{for} \quad 0 \leqslant k \leqslant 1 \\ S(\theta) &= kaE(\sqrt{(k^2-1)}/k, \theta) \qquad \text{for} \quad k \geqslant 1. \end{aligned} \tag{10}$$

However, it is the average Nusselt number that is more useful in engineering applications. The average Nusselt number, $\overline{Nu}$, is obtained by integrating equation (8)

$$\overline{Nu} = \frac{\bar{h}S}{k_e} = 4\left(\frac{-HF'(0)\,(1+k)}{2}\right)^{1/2} Pe_d^{1/2} \tag{11}$$

NOMENCLATURE

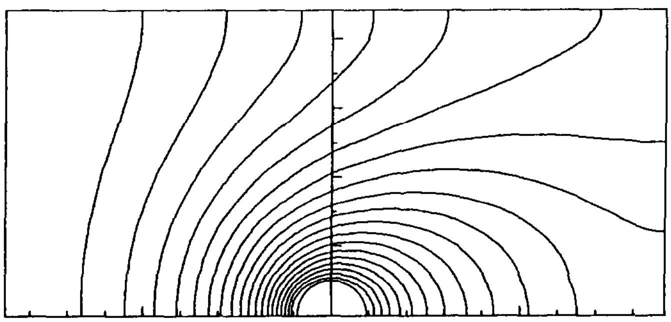
- a half length of elliptic axis oriented to stream direction [m]
- b half length of elliptic axis perpendicular to stream [m]
- E elliptic integral of the second kind
- F temperature profile function
- h heat transfer coefficient [$\mathrm{W\,m^{-2}\,K^{-1}}$]
- $\bar{h}$ average heat transfer coefficient [$\mathrm{W\,m^{-2}\,K^{-1}}$]
- H integral of F with respect to Δ
- k parameter defined by b/a
- k_e effective thermal conductivity [$\mathrm{W\,m^{-1}\,K^{-1}}$]
- L integral of U with respect to s
- Nu local Nusselt number, hs/k_e
- $\overline{Nu}$ average Nusselt number, $\bar{h}S/k_e$
- Pe Peclet number, Ux/α: where x is the characteristic length
- s coordinate of streamwise direction along the cylinder surface [m]
- S circumferential length of the cylinder cross-section [m]
- T temperature [K]
- ΔT temperature difference, $T_0 - T_\infty$ [K]
- U streamwise velocity on the cylinder surface [$\mathrm{m\,s^{-1}}$].

Greek symbols

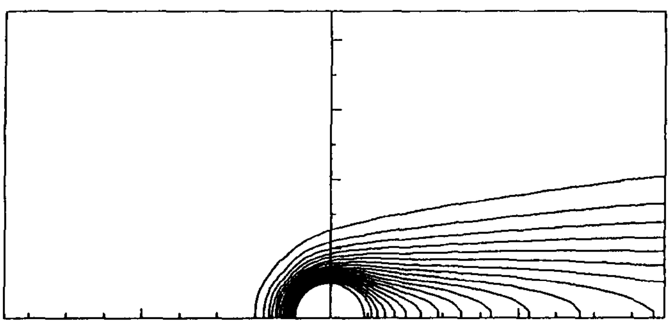
- α effective thermal diffusivity [$\mathrm{m^2\,s^{-1}}$]
- δ_T thermal boundary layer thickness [m]
- Δ non-dimensionalized coordinate of η
- η coordinate perpendicular to the cylinder surface [m]
- θ angle measured in streamwise direction [rad].

Subscripts

- $*$ dimensional quantities
- 0 conditions on the cylinder surface
- d characteristic length scale, $2a$ [m]
- ∞ conditions at infinity.



(a)



(b)

FIG. 2. Temperature field about a circular cylinder obtained by the complete two-dimensional solution ($k = 1$): (a) $Pe_d = 0.6$; (b) $Pe_d = 10$.

where S is the circumferential length which again can be expressed in terms of the elliptic integral

$$S = 4aE(\sqrt{(1-k^2)}, \pi/2) \quad \text{for} \quad 0 \leqslant k \leqslant 1$$
$$S = 4kaE(\sqrt{(k^2-1)}/k, \pi/2) \quad \text{for} \quad k \geqslant 1. \tag{12}$$

It is interesting to note that equation (11) with $k = 0$ corresponds to a heat transfer correlation over a flat plate [5]. The temperature profile cannot be determined uniquely in the present method. Table 1 shows how large the value of

Table 1. Choice of temperature profile function and its effect on the integral solution

$F(\Delta)$	H	$-F'(0)$	$\sqrt{(-HF'(0)/2)}$
$1-2\Delta+2\Delta^3-\Delta^4$	0.3	2	0.548
$1-\frac{3}{2}\Delta+\frac{1}{2}\Delta^3$	0.375	1.5	0.530
$1-\frac{4}{3}\Delta+\frac{1}{3}\Delta^4$	0.4	1.333	0.516
$1-\Delta$	0.5	1	0.5

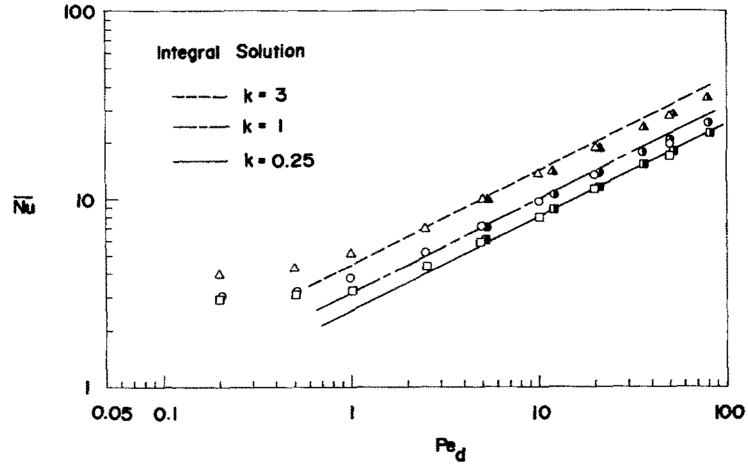


FIG. 3. The Pe_d dependency of $\overline{Nu}$ both by the integral solution and by the two-dimensional solution: □, ○, △ for $k = 0.25$, 1, 3, respectively, with 16 grid points on the elliptic arc and ◧, ◑, ▲ for $k = 0.25$, 1, 3, respectively, with 34 grid points.

$\sqrt{(-HF'(0)/2)}$ varies with temperature profile functions. It is found accordingly that the variation due to different choices is small enough to ensure the present integral solution. Incidentally Cheng [6] obtained 0.564 for the same coefficient in a circular cylinder case ($k = 1$).

3. TWO-DIMENSIONAL SOLUTION AND CONCLUDING REMARKS

So far we have developed an approximate solution by sacrificing the accuracy. In this section we provide the complete two-dimensional solution by means of finite differences to test the accuracy of the simplified solution and to clarify its range of applicability. We will not elaborate the numerical method in this note, since it is a fairly standard procedure today. The representative temperature fields obtained from the two-dimensional solution are shown in Figs. 2(a) and (b). With a small value of Pe_d it can be seen that the heat transfer mechanism is still largely dominated by heat conduction. On the other hand the temperature field changes drastically with the increase of Pe_d as it is shown in Fig. 2(b). The thermal boundary layer in this case is clearly defined and, therefore, one can expect that the integral solution yields good results in this regime.

We assemble the $\overline{Nu}$–Pe_d correlations for three different values of k in Fig. 3. Three lines indicate the results from the integral method, while six different symbols are obtained by the numerical method. Half filled symbols are generated by placing more grid points along the elliptic arc and its outer vicinity. It is seen that the lines and the symbols for each k agree well when $Pe_d > 2.5$. This confirms our earlier observations of the temperature fields. The agreement for $k = 0.25$ is excellent. As k increases, however, the average Nusselt numbers predicted by the integral method tend to exceed slightly the numerical results. Despite this fact the integral results still fall within a range of 15% off from the numerical ones in the presently investigated parametric values. The discrepancy may partly originate from the basic assumption made in developing the integral solution; the velocity within the thermal boundary layer is constant in η and equal to the slip velocity along the elliptic arc. The geometric bluntness to the flow, which becomes more pronounced with the increase of k, may eventually prohibit such a simplification on the velocity field. The temperature fields by the numerical solution with a large k also indicate that the area facing the incoming flow can be characterized quite differently from that behind the cylinder; the thermal boundary layer there does not grow as fast as the front with the increase of Pe_d.

In concluding this note it can be said that the compact expression for the average Nusselt number derived from the integral solution is satisfactory for most engineering purposes provided that the Peclet number $Pe_d > 2.5$. The critical Peclet number beyond which the integral solution is valid seems to decrease with increasing k. However, the accuracy of the integral solution at large Peclet numbers tends to deteriorate as k increases.

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